Non-equilibrium flow inside a wavy cylinder

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The small-perturbation theory for steady, inviscid, non-equilibrium flow is extended to include the transonic speed range. The resulting transonic small-perturbation equation for the velocity potential is solved for flow inside a wavy cylinder. It is shown that this solution gives a representation of transition through sonic speed at the narrowest sections of the cylinder. This extension of the theory is applied to non-equilibrium nozzle flow.

1. Introduction

An interesting non-equilibrium transonic-flow problem may arise in nozzles used in hypersonic shock tunnels. In some cases, the shock-heated gas may not have reached an equilibrium state before beginning its expansion through the nozzle. In order to study problems of this nature, a non-equilibrium transonic theory is necessary.

The linearized theory of non-equilibrium, steady and irrotational gas flows has been studied in considerable detail. Non-equilibrium processes considered in the theory are either single vibrational or chemical processes. In the former case, a new state variable T_i (the internal vibrational temperature) and in the latter case, α (the degree of dissociation for chemical non-equilibrium of a dissociating diatomic gas) are introduced in the analysis. A rate equation for either of these variables, which describes a small perturbation of the equilibrium flow condition, is taken in linear form since small deviations from an equilibrium state are considered. It has been shown (Vincenti 1959; Moore & Gibson 1960; Clarke 1960), that the perturbation velocity potential Φ in the axisymmetric case satisfies the equation

$$K\frac{\partial}{\partial x}\left(a\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right)+b\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}=0, \tag{1}$$

where K is a measure of the relaxation length; $a = 1-M_{f\infty}^2$, $b = 1-M_{e\infty}^2$; x is the axial and r the radial co-ordinate; and $M_{f\infty}$ and $M_{e\infty}$ are the frozen and equilibrium Mach numbers, based on the frozen and equilibrium speeds of sound $a_{f\infty}$ and $a_{e\infty}$, respectively.

Solutions obtained by Vincenti (1959) and Clarke (1961) with equation (1) for flow about bodies show several interesting properties. For example, if a finite value is assigned to the reaction-length parameter K, the solution for flow past a body does not break down in the transonic speed region. However, although finite values are obtained for all flow variables, the solutions in this region suffer shortcomings similar to those of solutions to the classical Prandtl–Glauert equation. In order to obtain a valid solution at transonic Mach numbers, the terms obviously missing in equation (1) have to be included.

In this paper, the correct form of equation (1) in the transonic speed range will be given under the assumption that the rate law still may be given in linear form at transonic speeds. Then, in order to exhibit the essential physical properties of non-equilibrium nozzle flows, a simple solution to the transonic equation will be developed. Such a solution, involving comparatively simple mathematical formalism, is obtained by considering flow inside a wavy cylinder. Subsequently, this solution will be used for a qualitative study of non-equilibrium versus equilibrium flow phenomena in a nozzle.

2. Transonic correction of the governing equation

In order to derive the correct small-disturbance transonic equation, one ought to start with the exact gas-dynamic equations; however, one can also reason by analogy with the classical theory developed in correcting the Prandtl–Glauert equation in the transonic flow region. From this analysis, it is known that, at transonic speeds, the Prandtl factor $1-M_\infty^2$ must be replaced by $1-M^2$, and that, upon subsequent series development of this Mach number factor in powers of Φ_x, the additional term of importance is found (Oswatitsch 1956).

By applying this reasoning to the present case, $1-M_{f\infty}^2$ and $1-M_{e\infty}^2$ should be replaced by $1-M_f^2$ and $1-M_e^2$ in equation (1). Upon developing these factors in powers of Φ_x, we obtain

$$1-M_f^2 = 1-M_{f\infty}^2 - \frac{1-M_{f\infty}^2}{(1/M_{f\infty}^*)-1}\frac{\Phi_x}{U_\infty}+\dots, \tag{2}$$

and

$$1-M_e^2 = 1-M_{e\infty}^2 - \frac{1-M_{e\infty}^2}{(1/M_{e\infty}^*)-1}\frac{\Phi_x}{U_\infty}+\dots, \tag{3}$$

where $M_{f\infty}^*$ and $M_{e\infty}^*$ are the critical Mach numbers based on the frozen and equilibrium critical speeds of sound, respectively. By introducing factors A and B defined as

$$A = \frac{1-M_{f\infty}^2}{(1/M_{f\infty}^*)-1}, \quad B = \frac{1-M_{e\infty}^2}{(1/M_{e\infty}^*)-1}, \tag{4}$$

the small-disturbance, transonic, non-equilibrium equation is obtained:

$$K(a\Phi_{xxx}+\Phi_{xrr}+r^{-1}\Phi_{xr})+b\Phi_{xx}+\Phi_{rr}+r^{-1}\Phi_r = \frac{KA}{U_\infty}(\Phi_x\Phi_{xx})_x+\frac{B}{U_\infty}\Phi_x\Phi_{xx}. \tag{5}$$

Equation (5) is, of course, the consequence of a postulated linear rate equation, which is assumed to be valid regardless of the particular range of speed considered. This rate equation thus predicts small deviations from an equilibrium state of the new state variable, which is either T_i or α as before. In the small perturbation theory, compressibility effects on the flow field are profound around Mach 1, which necessitates the delinearization of the velocity in equation (1). However, the influence of the rate process on the flow field, relatively speaking, is equally sensitive to density changes in any particular speed range.

Therefore, the assumption of linearity of the rate equation in the transonic range seems to have the same physical justification as it would in any other speed régime.

Equation (5) is highly non-linear since the right-hand member contains products of derivatives of Φ. By letting $K = 0$, the equation reduces to the classical small-disturbance transonic equation for equilibrium flow. At the other limit, i.e. $K \to \infty$, the equation is essentially the classical small-disturbance transonic equation, but with the Mach number based on the frozen speed of sound.

Because of the non-linearity of the transonic equation, some form of linearization is generally employed in order to obtain solutions for actual flow problems. One method of linearization, in nozzle flows (Behrbohm 1950), as well as in flows over bodies (Oswatitsch & Keune 1955), is to replace Φ_{xx} in the non-linear term by a constant. This procedure introduces a certain arbitrariness into the analysis since the value of the constant cannot be obtained from the analysis itself. Nevertheless, useful and simple results are obtained, which, with a judicious choice of the constant value of Φ_{xx}, will give a good approximation to the flow field.

By following these arguments in our analysis, we accordingly let

$$C = \Phi_{xx}/U_\infty = \text{const.} \tag{6}$$

in the non-linear terms on the right-hand side of equation (5), and obtain

$$K(a\Phi_{xxx} + \Phi_{xrr} + r^{-1}\Phi_{xr}) + (b - KAC)\,\Phi_{xx} + \Phi_{rr} + r^{-1}\Phi_r - BC\Phi_x = 0. \tag{7}$$

We shall calculate C by using the one-dimensional analysis for flow through converging-diverging channels. It is known that the velocity gradient dW/dx at the critical section in such a channel is related to the cross-sectional area f in the following way (Oswatitsch 1956):

$$\frac{f^*}{a^*}\frac{dW}{dx} = \left[\frac{f^*}{\gamma+1}\left(\frac{d^2f}{dx^2}\right)_{f=f^*}\right]^{\frac{1}{2}}, \tag{8}$$

where the asterisk denotes the critical value at the narrowest section of the channel, and γ is the ratio of the specific heats. By substituting CU_∞ for the gradient dW/dx, and a_f^*, for a^* in equation (8), we obtain

$$C = \frac{1}{f^* M_{f\infty}^*}\left[\frac{f^*}{\gamma+1}\left(\frac{d^2f}{dx^2}\right)_{f=f^*}\right]^{\frac{1}{2}}. \tag{9}$$

In the case where $K = 0$, $M_{f\infty}^*$ must be replaced by $M_{e\infty}^*$ in the formula above.

3. Flow inside a wavy cylinder

We shall now find the solution to equation (7) for flow through the narrowest sections of a wavy cylinder. Subsequently, it will be seen that, in the transonic speed range, such a flow will exhibit all the physical phenomena of interest. The technique used by Vincenti (1959) will be employed in developing the solution.

Let the wavy cylinder wall be given by

$$r = R + \tau \sin 2\pi x/L, \tag{10}$$

where R is the mean radius, τ the amplitude, and L the wavelength (see figure 1). After introducing the new independent variables ξ and η defined by

$$\xi = 2\pi x/L, \quad \eta = 2\pi r/L, \tag{11}$$

and letting

$$k = 2\pi K/L, \quad c = CL/2\pi, \tag{12}$$

equation (7) takes the form

$$k(a\Phi_{\xi\xi\xi} + \Phi_{\xi\eta\eta} + \eta^{-1}\Phi_{\xi\eta}) + (b - kAc)\,\Phi_{\xi\xi} + \Phi_{\eta\eta} + \eta^{-1}\Phi_{\eta} - Bc\Phi_{\xi} = 0. \tag{13}$$

The value of c in equation (13) is found from equations (9), (10) and (12),

$$c = \frac{1}{M_f^*}\left(\frac{2}{\gamma+1}\,\frac{\tau}{R-\tau}\right)^{\frac{1}{2}}. \tag{14}$$

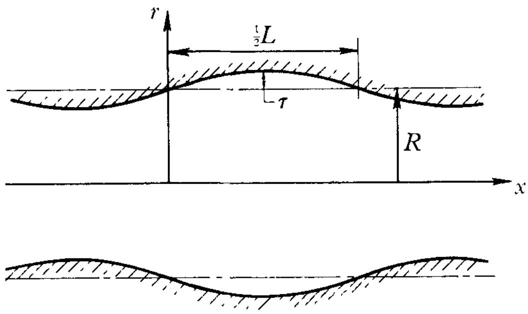

FIGURE 1. Channel geometry.

The tangency condition at the wall, taken in the usual simplified form, gives the following boundary condition to supplement equation (13):

$$\Phi_{\eta} = U_{\infty}\tau\cos\xi \quad \text{on} \quad \eta = 2\pi R/L. \tag{15}$$

A fundamental solution to equation (13) is

$$\Phi(\xi, \eta) = e^{\alpha\xi}F(\eta), \tag{16}$$

where α is in general a complex number. By inserting equation (16) into equation (13), the following ordinary differential equation is obtained for $F(\eta)$:

$$F'' + \eta^{-1}F' - \beta^2 F = 0, \tag{17}$$

with
$$\beta^2 = -\{ka\alpha^3 + (b - kAc)\,\alpha^2 - Bc\alpha\}/(1 + k\alpha). \tag{18}$$

The general solution to equation (17) is a combination of modified Bessel functions of the first and second kind, of order zero (Erde'lyi *et al.* 1953):

$$F(\eta) = C_1 I_0(\beta\eta) + C_2 K_0(\beta\eta). \tag{19}$$

Since only the internal flow of the cylinder is of interest and because the solution must be regular for $\eta = 0$, the constant C_2 is zero. Furthermore, since we are seeking a periodic solution, Φ may be written in the form

$$\Phi(\xi, \eta) = C_1 e^{\alpha\xi} I_0(\beta\eta) + \bar{C}_1 e^{\bar{\alpha}\xi} I_0(\bar{\beta}\eta), \tag{20}$$

where the barred quantities are the complex conjugates of the unbarred ones. In particular, the solution must be periodic in ξ; therefore, we choose $\alpha = i$ and $\bar{\alpha} = -i$ in equation (20). Also, by letting $\beta = \delta + i\lambda$ and $\bar{\beta} = \delta - i\lambda$, the solution takes the form

$$\Phi(\xi, \eta) = C_1(\cos\xi + i\sin\xi)\, I_0(\delta\eta + i\lambda\eta) + \bar{C}_1(\cos\xi - i\sin\xi)\, I_0(\delta\eta - i\lambda\eta). \quad (21)$$

To find the real part of equation (21) some theorems from the theory of Bessel functions will be used. For all z, t, we have (Erde'lyi *et al.* 1953):

$$I_0(z+t) = I_0(z)\, I_0(t) + 2\sum_{n=1}^{\infty} I_n(z)\, I_n(t). \quad (22)$$

Furthermore, by using the formula

$$I_n(iz) = e^{\frac{1}{2}n\pi i} J_n(z), \quad (23)$$

where $J_n(z)$ is the Bessel function of the first kind, we obtain

$$I_0(\delta\eta + i\lambda\eta) = M(\eta) + iN(\eta), \quad (24)$$

where
$$M(\eta) = J_0(\delta\eta)\, I_0(\lambda\eta) - 2\sum_{n=0}^{\infty} (-1)^n J_{2n+2}(\delta\eta)\, I_{2n+2}(\lambda\eta), \quad (25)$$

and
$$N(\eta) = 2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\delta\eta)\, I_{2n+1}(\lambda\eta). \quad (26)$$

Similarly, with the help of the relation

$$I_n(-z) = e^{i\pi n} I_n(z), \quad (27)$$

and equations (23) and (22), we get

$$I_0(\delta\eta - i\lambda\eta) = M(\eta) - iN(\eta). \quad (28)$$

The expression for Φ now takes the real form

$$\begin{aligned} \Phi &= C_1(\cos\xi + i\sin\xi)\,(M + iN) + \bar{C}_1(\cos\xi - i\sin\xi)\,(M - iN) \\ &= D(M\cos\xi - N\sin\xi) + E(M\sin\xi + N\cos\xi), \end{aligned} \quad (29)$$

where D and E are real constants related to C_1 and $\bar{C}_1$ through the relations

$$D = C_1 + \bar{C}_1, \quad E = i(C_1 - \bar{C}_1). \quad (30)$$

The two parameters δ and λ can be found by substituting $\beta = \delta + i\lambda$ and $\alpha = i$ into equation (18). By equating real and imaginary parts, the following two equations are obtained:

$$\delta^2 - \lambda^2 = \frac{b + ak^2 + kc(B-A)}{1+k^2} = P, \quad 2\delta\lambda = \frac{k(a-b) + c(B + Ak^2)}{1+k^2} = Q. \quad (31a, b)$$

Since $a_{f\infty} > a_{e\infty}$ (which means that $M_{f\infty} < M_{e\infty}$), $a > b$ and $B > A > 0$. Thus, since $c > 0$, we always have $Q > 0$. However, P may be either positive or negative depending on the values of $M_{f\infty}$, $M_{e\infty}$, k, and c. The formal solutions for δ and λ are

$$\delta = \pm[\tfrac{1}{2}\{P \pm (P^2+Q^2)^{\frac{1}{2}}\}]^{\frac{1}{2}}, \quad \lambda = \pm[\tfrac{1}{2}\{-P \pm (P^2+Q^2)^{\frac{1}{2}}\}]^{\frac{1}{2}}. \quad (32a, b)$$

Both δ and λ must be real, and therefore the minus sign in front of $(P^2+Q^2)^{\frac{1}{2}}$ in equations (32*a*) and (32*b*) has to be discarded. To satisfy equation (31*b*), δ and λ must be either both positive or both negative. In deriving the formula for Φ, however, it was tacitly assumed that the functions $M(\eta)$ and $N(\eta)$ have positive arguments. The positive signs for δ and λ are therefore chosen. The final formulas for δ and λ are

$$\left.\begin{matrix}\delta\\ \lambda\end{matrix}\right\} = \left\{\frac{1}{2(1+k^2)}\left[\pm(b-kAc+k[ak+cB])+\{(1+k^2)\left[(ak+cB)^2+(b-kAc)^2\right]\}^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}, \tag{33}$$

where the upper sign goes with δ and the lower with λ.

The boundary condition given by equation (15) must also be satisfied. By taking the derivative of equation (29) with respect to η, we obtain

$$\Phi_\eta = D(M'\cos\xi - N'\sin\xi) + E(M'\sin\xi + N'\cos\xi), \tag{34}$$

where, with help of the formulas (Erde'lyi *et al.* 1953)

$$2J_n'(z) = J_{n-1}(z) - J_{n+1}(z), \tag{35}$$

$$2I_n'(z) = I_{n-1}(z) + I_{n+1}(z), \tag{36}$$

we have, from equations (25) and (26),

$$\begin{aligned} M'(\eta) = \frac{dM}{d\eta} &= \lambda I_1(\lambda\eta)\,J_0(\delta\eta) - \delta I_0(\lambda\eta)\,J_1(\delta\eta) - \sum_{n=0}^{\infty}(-1)^n \\ &\times\{\lambda J_{2n+2}(\delta\eta)\,[I_{2n+1}(\lambda\eta)+I_{2n+3}(\lambda\eta)] + \delta I_{2n+2}(\lambda\eta)\,[J_{2n+1}(\delta\eta) - J_{2n+3}(\delta\eta)]\}, \end{aligned} \tag{37}$$

and
$$\begin{aligned} N'(\eta) = \frac{dN}{d\eta} = \sum_{n=0}^{\infty}(-1)^n\{&\lambda J_{2n+1}(\delta\eta)\,[I_{2n}(\lambda\eta)+I_{2n+2}(\lambda\eta)] \\ &+\delta I_{2n+1}(\lambda\eta)\,[J_{2n}(\delta\eta)-J_{2n+2}(\delta\eta)]\}. \end{aligned} \tag{38}$$

By inserting the value $\eta = 2\pi R/L$ into equations (37) and (38) and using equations (34) and (15), we obtain simultaneous equations for D and E,

$$U_\infty\tau = DM_0' + EN_0', \quad 0 = -DN_0' + EM_0', \tag{39a, b}$$

where M_0' and N_0' stand for $M'(2\pi R/L)$ and $N'(2\pi R/L)$, respectively. The solutions for D and E are

$$D = \frac{U_\infty\tau M_0'}{M_0'^2+N_0'^2}, \quad E = \frac{U_\infty\tau N_0'}{M_0'^2+N_0'^2}. \tag{40}$$

The potential Φ can now be written in its final form

$$\Phi = U_\infty\tau[(MM_0'+NN_0')\cos\xi + (MN_0'-NM_0')\sin\xi]/(M_0'^2+N_0'^2). \tag{41}$$

4. The velocity field and the parameters δ and λ

In order to study various properties of the flow field, we next derive the disturbance velocities Φ_x and Φ_r, and obtain from equation (41)

$$\frac{u}{U_\infty} = \frac{\Phi_x}{U_\infty} = \frac{2\pi\tau/L}{M_0'^2+N_0'^2}\{[M(\eta)\,N_0' - N(\eta)\,M_0']\cos\xi - [M(\eta)\,M_0' + N(\eta)\,N_0']\sin\xi\}, \tag{42}$$

$$\frac{v}{U_\infty} = \frac{\Phi_r}{U_\infty} = \frac{2\pi\tau/L}{M_0'^2+N_0'^2}\{[M'(\eta)\,M_0' + N'(\eta)\,N_0']\cos\xi + [M'(\eta)\,N_0' - N'(\eta)\,M_0']\sin\xi\}. \tag{43}$$

The potential expression (41) and its derivatives (42) and (43) are very general in the sense that they cover all reacting and non-reacting flows inside a wavy cylinder, from sub- to supersonic Mach numbers. The parameters δ and λ take different values for each flow case. For example, a classical equilibrium ($k = 0$) transonic flow has the parameters (see equation (33))

$$\left.\begin{matrix}\delta\\ \lambda\end{matrix}\right\} = 2^{-\frac{1}{2}}\left\{\pm(1-M_{e\infty}^2)+\left[(1-M_{e\infty}^2)^2\left(1+\frac{2}{\gamma+1}\frac{\tau}{R-\tau}\frac{1}{(1-M_{e\infty}^{*})^2}\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}. \quad (44)$$

When $M_{e\infty} \to 1$, this expression reduces to the form

$$\delta = \lambda = \left[\tfrac{1}{2}(\gamma+1)\frac{\tau}{R-\tau}\right]^{\frac{1}{4}}. \quad (45)$$

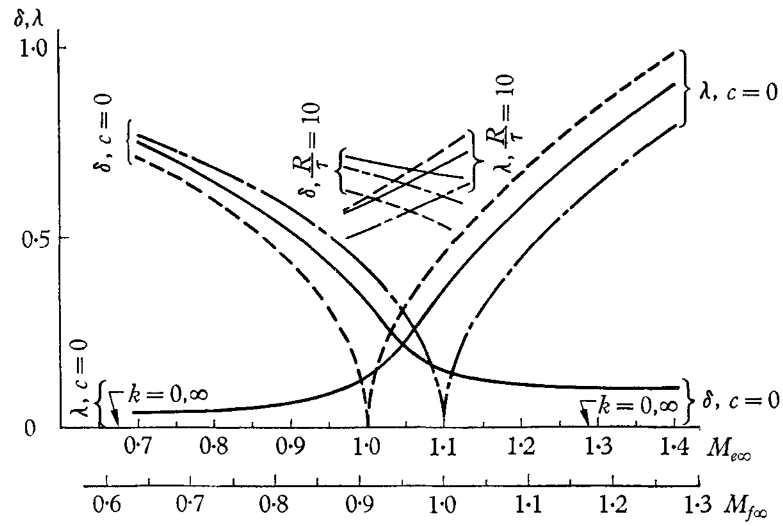

FIGURE 2. δ and λ as a function of Mach number and with k and R/τ as parameters. ———, $k = 1$; — — —, $k = \infty$; - - -, $k = 0$.

For a pure sub- or supersonic flow, c equals zero, since for $c = 0$ equation (7) reduces to the form of equation (1). To illustrate this, if subsonic equilibrium ($k = 0$) flow is considered, δ and λ are given by

$$\delta = (1-M_{e\infty}^2)^{\frac{1}{2}} \quad (\lambda = 0),$$

the potential Φ in this case reduces, as it should, to a known form in classical theory (Howarth 1953).

In order to exhibit the overall properties of δ and λ in greater detail, δ and λ have been plotted in figure 2 as functions of Mach number for a ratio of $a_{f\infty}/a_{e\infty}$ of 11/10, and with k and R/τ (or c) as parameters. For k, the values 0, 1, and ∞ were chosen, representing equilibrium, non-equilibrium and frozen flows, respectively. The value $k = 1$ for a non-equilibrium situation was chosen because it was found that this value of k gave very nearly maximum deviations from equilibrium and frozen values of δ and λ in regions where δ and/or λ are small.

In the transonic region, i.e. the region defined essentially by $M_{f\infty} < 1$ but $M_{e\infty} > 1$, the various curves for different values of R/τ overlap considerably.

Therefore, only one set of curves for a finite value of R/τ is shown ($R/\tau = 10$). For comparison, the values of δ and λ for $c = 0$ were extended through the transonic region. It appears, however, that, in the transonic region, the curves labelled $c = 0$ are only strictly applicable to the case where the value of R/τ is great, actually only for $R/\tau = \infty$.

Consider next a sequence of flows through the channel with increasing Mach number starting from the subsonic side. At first, the values of δ and λ are those pertaining to $c = 0$. Upon approaching the transonic region, a transition must take place from the curves for which $c = 0$ to the curves that are dependent on the value of R/τ and valid in the transonic region. A similar transition occurs again upon leaving the transonic region; hence, for pure supersonic Mach numbers, δ and λ have values that are obtained by letting $c = 0$. The present theory with its crude assumptions cannot predict these transitions, although the values of δ and λ themselves, within the framework of the linearization, are correctly given in the transonic as well as in the sub- and supersonic regions.

It should be pointed out that in Vincenti's paper (1959) treating the flow over a wavy wall, parameters δ and λ also appear, having the same meaning as here. The expressions for δ and λ found here are identical to those found by Vincenti if, in the present formulas, $c = 0$.

5. A transonic flow example

In order to study the effect of relaxation on the flow, two actual flow patterns in the transonic region have been computed using the same geometrical configuration and the same values of a and b but different values for the relaxation-length parameter k. The channel is characterized by the parameters R, L and τ, and in the computed examples $R/L = 0{\cdot}5$ and $\tau/L = 0{\cdot}1$. This gives a nozzle for which the ratio of the radius of curvature of the wall l to the radius of the cross-section at the throat is $l/(R-\tau) = 0{\cdot}634$. The values $a = 0{\cdot}1$ and $b = -0{\cdot}1$ were chosen to correspond to the case where $M_{f\infty} < 1$ and $M_{e\infty} > 1$. The ratio $a_{f\infty}/a_{e\infty}$ is then very nearly equal to $1{\cdot}1$.

The intermediate value of $k = 1$ was chosen in one of the examples as being a value for which maximum relaxation effects can be expected according to figure 2. The results of the flow field computation for $k = 1$ using equations (42) and (43) are shown in figure 3, where the solid lines represent constant speed lines, $[(U_\infty + u)^2 + v^2]^{\frac{1}{2}}/a^*_{f\infty}$. The absolute value of the velocity vector is made dimensionless by the use of the physically significant parameter $a^*_{f\infty}$ in this non-equilibrium case.

The equilibrium ($k = 0$) flow pattern is also shown in figure 3 (dotted lines). For this case, the magnitude of the velocity vector is made dimensionless by the use of $a^*_{e\infty}$, the significant parameter for equilibrium flow. The correct physical behaviour of the flow fields is thus shown in figure 3. It is seen that the sonic line in the equilibrium case has been moved somewhat upstream as compared with the non-equilibrium case, but in other respects has nearly the same form. The constant speed lines in both cases resemble familiar forms found in nozzle-flows, and the transition through sonic speed is clearly shown.

The value of Φ_{xx} is found to remain almost constant in the flow field for a fixed value of η and M_f^*, satisfying the inequality $0{\cdot}9 < M_f^* < 1{\cdot}1$. The value of $(L/2\pi)(\Phi_{xx}/U_\infty)$ in this Mach-number range varies almost linearly with η from $0{\cdot}24$ at $\eta = 0$ to $0{\cdot}68$ at $\eta = \pi$. The mean value of these two numbers corresponds very closely to the value of c computed by using equation (14).

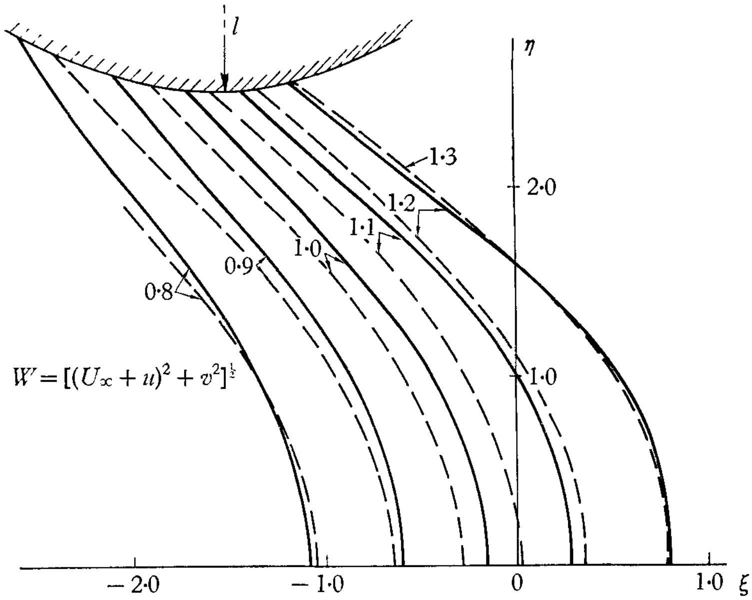

FIGURE 3. Transition through sonic speed in equilibrium and non-equilibrium flow. ———, $W/a_f^*(k = 1)$; – – –, $W/a_e^*(k = 0)$.

The change in other variables in the transonic region of the flow can readily be obtained from our solution. One finds, for example, that the disturbance value of α (or T_i) in the non-equilibrium case reaches a maximum value on a line that coincides with the sonic line at the axis, but otherwise lies slightly upstream of the sonic line.

6. Final remarks

The solution for non-equilibrium transonic flow through the narrowest sections of a wavy cylinder has been shown to give a representation of transition through sonic speed. Some physical properties of non-equilibrium nozzle flow may, therefore, be studied with this solution and compared with a corresponding equilibrium flow in order to obtain the effects of relaxation phenomena in the nozzle. The equilibrium or frozen solutions are presented as special cases of the more general solution. It is of interest that the sonic line in non-equilibrium flow is slightly downstream, as compared with its counterpart in equilibrium flow.

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